# October Math Gems

PROBLEM OF THE WEEK 15

## §1 Problems

**Problem 1.1.** Show that in any triangle $ABC$ we have $\frac{a\cos C - b\cos B}{a\cos B - b\cos A} + \cos C = 0$

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**Problem 1.2.** Show that in any triangle $ABC$ $\sin\frac{A}{2} \leq \frac{a}{2\sqrt{bc}}$

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**Problem 1.3.** If $\tan A = \frac{1}{2m+1}$ and $\tan B = \frac{m}{m+1}$ where $A$ and $B$ then $A + B$ is equal to

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**Problem 1.4.** If $a$ is an acute angle and $x$ is $(+)$ and $2x\sin^2\frac{a}{2} + 1 = x$ then, $\tan a =$ (with respect to x)

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**Problem 1.5.** Let $f : R- > R$ be defined as $f(x) = 2x + |x|$, then $f(2x) + f(-x) - f(x) =$

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**Problem 1.6.** If $f(x) = \frac{x}{\sqrt{1+x^2}}$ ,then $(fofof)(x) =$

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**Problem 1.7.** If $f(x) = \log(\frac{1+x}{1-x})$ and $g(x) = \frac{3x+x^3}{1+3x^2}$ ,then $fog(x) =$

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**Problem 1.8.** Let $x$ be a real number such that $\sin^{10}x + \cos^{10}x = \frac{11}{36}$. Then $\sin^{12}x + \cos^{12}x = \frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

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**Problem 1.9.** In triangle $ABC$, point $D$ divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let $E$ be the midpoint of $\overline{BD}$ and let $F$ be the point of intersection of line $BC$ and line $AE$. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

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**Problem 1.10.** Find all real numbers $x$ that satisfy the equation

$$\frac{x-2020}{1} + \frac{x-2019}{2} + \cdots + \frac{x-2000}{21} = \frac{x-1}{2020} + \frac{x-2}{2019} + \cdots + \frac{x-21}{2000},$$

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**Problem 1.11.** Let $f(x) = \frac{ax+b}{cx+d}$ ,and $f \circ f(x) = x$
For $a = -22$ ,then $d =$

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**Problem 1.12.** Find all $x, y \in \left(0, \frac{\pi}{2}\right)$ such that

$$\frac{\cos x}{\cos y} = 2\cos^2 y,$$
$$\frac{\sin x}{\sin y} = 2\sin^2 y.$$

Then, prove that $x = y$

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**Problem 1.13.** If it were two hours later, it would be half as long until midnight as it would be if it were an hour later than it is now. What time is it now?

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**Problem 1.14.** Given the following system of equations

$$(x+y)^2 - z^2 = 4$$

$$(y+z)^2 - x^2 = 9$$

$$(z+x)^2 - y^2 = 36$$

Then, find $x+y+z$

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**Problem 1.15.** Evaluate

$$\left(\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{100^3-1}{100^3+1}\right)$$

The expression can be written as $\frac{a}{b}$ where $a$ and $b$ are relatively co-prime integers. Find $a+b$

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**Problem 1.16.** Solve for $x$

$$4^x + 9^x + 49^x = 6^x + 14^x + 21^x$$

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**Problem 1.17.**

$$k\left(x+\frac{x}{2}\right) = \frac{x^3+1}{x} + \frac{x^3+8}{2x^2} + 3$$

Find $k(1)$

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**Problem 1.18.** Solve the inequality

$$\sqrt{x-1} + \sqrt{x^2-1} < \sqrt[3]{x^3}$$

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**Problem 1.19.** Evaluate the following expression

$$\left(\frac{x\sqrt{x}}{\sqrt{1-x^3}} + \frac{\sqrt{1-x^3}}{x\sqrt{x}}\right)^{-1}$$

if

$$x = \sqrt[3]{\frac{a-\sqrt{a^2-b^2}}{2a}}$$

Where $a, b \in R$ are such that $0 <\mid a \mid\geq\mid b \mid> 0$

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**Problem 1.20.**

$$a + \frac{1}{b+\frac{1}{c+\frac{1}{d}}} = \frac{29}{12}$$

Then, $a^3+b^3+c^3+d^3$

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